Graphs with small diameter determined by their D-spectra

Jie Xue[a]   Ruifang Liu[a*]

[a] School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, Henan 450001, China

Abstract

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The distance matrix $D(G) = (d_{ij})_{n \times n}$ is the matrix indexed by the vertices of G, where d_{ij} denotes the distance between the vertices v_i and v_j. Suppose that $\lambda_1(D) \geq \lambda_2(D) \geq \cdots \geq \lambda_n(D)$ are the distance spectrum of G. The graph G is said to be determined by its D-spectrum if with respect to the distance matrix $D(G)$, any graph having the same spectrum as G is isomorphic to G. In this paper, we give the distance characteristic polynomial of some graphs with small diameter, and also prove that these graphs are determined by their D-spectra.

1 Introduction

All graphs considered here are simple, undirected and connected. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Two vertices u and v are called adjacent if they are connected by an edge, denoted by $u \sim v$. Let $N_G(v)$ denote the neighbor set of v in G. The degree of a vertex v, written by $d_G(v)$ or $d(v)$, is the number of edges incident with v. Let X and Y be subsets of vertices of G. The induced subgraph $G[X]$ is the subgraph of G whose vertex set is X and whose edge set consists of all edges of G which have both ends in X. We denote by $E[X, Y]$ the set of edges of G with one end in X and the other end in Y, and denote by $e[X, Y]$ their number. The distance between vertices u and v of a graph G is denoted by $d_G(u, v)$. The diameter of G, denoted by $\mathrm{diam}(G)$, is the maximum distance between any pair of vertices of G.

The complete product $G_1 \bigtriangledown G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 to every vertex of G_2. Denote by K_n, C_n, P_n and S_n the complete graph, the cycle, the path and the star, respectively, each on n vertices. Let K_n^c denote the complement of K_n.

The distance matrix $D(G) = (d_{ij})_{n \times n}$ of a connected graph G is the matrix indexed by the vertices of G, where d_{ij} denotes the distance between the vertices v_i and v_j. Let $\lambda_1(D) \geq \lambda_2(D) \geq \cdots \geq \lambda_n(D)$ be the spectrum of $D(G)$, that is, the distance spectrum of G. The polynomial $P_D(\lambda) = \det(\lambda I - D(G))$ is defined as the distance characteristic polynomial of a graph G. Two graphs are said to be D-cospectral if they have the same distance spectrum. A graph G is said to be determined by its D-spectrum if there is no other non-isomorphic graph D-cospectral to G.

Which graphs are determined by their spectrum seems to be a difficult and interesting problem in the theory of graph spectra. This question was proposed by Dam and Haemers in [3]. In this paper, Dam and Haemers investigated the cospectrality of graphs up to order 11. They showed that the adjacency matrix appears to be the worst representation in terms of producing a large number of cospectral graphs. The Laplacian is superior in this regard and the signless Laplacian even better. Later, Dam et al. [4, 5] provided two excellent surveys on this topic.

Up to now, only a few families of graphs were shown to be determined by their spectra, most of which were restricted to the adjacency, Laplacian or signless Laplacian spectra. In particular, there are much fewer results on which graphs are determined by their D-spectra. In [7], Lin et al. proved that the complete graph K_n, the complete bipartite graph K_{n_1,n_2} and the complete split graph $K_a \bigtriangledown K_b^c$ are determined by their D-spectra, and the authors also conjecture that the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is determined by its D-spectrum. Recently, Jin and Zhang [6] have confirmed the conjecture.

In this paper, we consider four kinds of graphs K_n^h, K_n^{s+t}, $K_n^{s,t}$ and the friendship graph F_n^k in Fig. 1.

- K_n^h: the graph obtained by attaching $n - h$ pendant edges to a vertex of K_h.
- K_n^{s+t}: the graph obtained by adding one edge joining a vertex of K_s to a vertex of K_t.
- $K_n^{s,t}$: the graph obtained by identifying a vertex of K_s and a vertex of K_t.
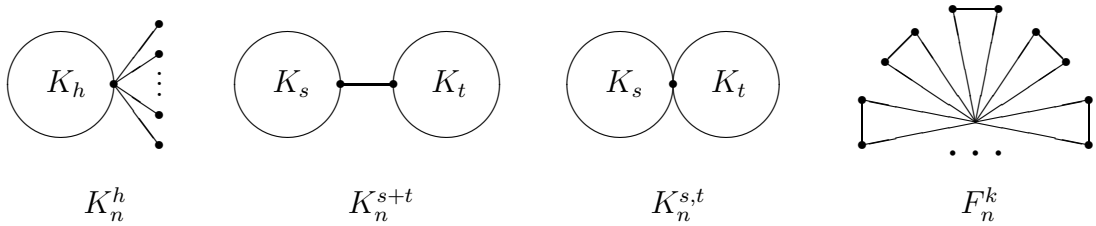- F_n^k: the graph obtained by joining k copies of the cycle C_3 with a common vertex.

Fig. 1. Graphs K_n^h, K_n^{s+t}, $K_n^{s,t}$ and F_n^k.

Zhang et al. [10] proved the graph K_n^h and its complement are determined by their adjacency spectra, and by their Laplacian spectra. In [1], Abdollahi et al. proved the complement of the friendship graph is determined by its adjacency spectrum. In this paper, we show that these four kinds of graphs K_n^h, K_n^{s+t}, $K_n^{s,t}$ and F_n^k are determined by their D-spectra.

Clearly, it is not our concerns if the above graphs are isomorphic to complete graphs or complete bipartite graphs. For K_n^h, it is a complete bipartite graph if $h = 1$ or $h = 2$, and is a complete graph if $n = h$. When $s = 1$ or $t = 1$, K_n^{s+t} and K_n^h are isomorphic. When $s = 1$ or $t = 1$, $K_n^{s,t}$ is a complete graph. Hence for the convenience of discussion, we have the following agreement: K_n^h ($h \geq 3, n \geq h + 1$), K_n^{s+t} ($n = s + t, s \geq 2, t \geq 2$), $K_n^{s,t}$ ($n = s + t - 1, s \geq 2, t \geq 2$) and F_n^k ($n = 2k + 1, k \geq 2$).

2 Preliminaries

In this section, we give some useful lemmas and results. The following lemma is well-known Cauchy Interlace Theorem.

Lemma 2.1 ([2]) *Let A be a Hermitian matrix of order n with eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$, and B be a principal submatrix of A of order m with eigenvalues $\mu_1(B) \geq \mu_2(B) \geq \cdots \geq \mu_m(B)$. Then $\lambda_{n-m+i}(A) \leq \mu_i(B) \leq \lambda_i(A)$ for $i = 1, 2, \ldots, m$.*

Applying Lemma 2.1 to the distance matrix D of a graph, we have

Lemma 2.2 *Let G be a graph of order n with distance spectrum $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, and H be an induced subgraph of G on m vertices with the distance spectrum $\mu_1(H) \geq \mu_2(H) \geq \cdots \geq \mu_m(H)$. Moreover, if $D(H)$ is a principal submatrix of $D(G)$, then $\lambda_{n-m+i}(G) \leq \mu_i(H) \leq \lambda_i(G)$ for $i = 1, 2, \ldots, m$.*

Lemma 2.3 ([7]) *Let G be a connected graph and D be the distance matrix of G. Then $\lambda_n(D) = -2$ with multiplicity $n - k$ if and only if G is a complete k-partite graph for $2 \leq k \leq n - 1$.*

Lemma 2.4 ([9]) *Let G be a graph with order n and $diam(G) = 2$. If G' has the same distance spectrum as G, then*

•$|E(G)| = |E(G')|$ *when* $diam(G') = 2$;

•$|E(G)| < |E(G')|$ *when* $diam(G') \geq 3$.

Theorem 2.5 *Let $3 \leq h \leq n - 1$. The distance characteristic polynomial of K_n^h is*

$$P_D(\lambda) = (\lambda+1)^{h-2}(\lambda+2)^{n-h-1}[\lambda^3+(h+4-2n)\lambda^2+(5-2h-2nh+2h^2-n)\lambda-nh+h^2-2h+2].$$

3

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the distance spectrum of K_n^h. Then
- $\lambda_1 > 0$, $-1 < \lambda_2 < -\frac{1}{2}$ and $\lambda_3 = -1$.
- $\lambda_{n-1} \in \{-1, -2\}$ and $\lambda_n < -2$.

Proof. It is clear that the diameter of K_n^h is 2, and the distance matrix of K_n^h is

$$
D = \begin{pmatrix}
0 & \cdots & 1 & 1 & 2 & \cdots & 2 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \cdots & 0 & 1 & 2 & \cdots & 2 \\
1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\
2 & \cdots & 2 & 1 & 0 & \cdots & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
2 & \cdots & 2 & 1 & 2 & \cdots & 0
\end{pmatrix}.
$$

Then

$$
\det(\lambda I - D) = \begin{vmatrix}
\lambda & \cdots & -1 & -1 & -2 & \cdots & -2 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & \cdots & \lambda & -1 & -2 & \cdots & -2 \\
-1 & \cdots & -1 & \lambda & -1 & \cdots & -1 \\
-2 & \cdots & -2 & -1 & \lambda & \cdots & -2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-2 & \cdots & -2 & -1 & -2 & \cdots & \lambda
\end{vmatrix}
$$

$$
= \begin{vmatrix}
\lambda & -1 & \cdots & -1 & -1 & -2 & -2 & \cdots & -2 \\
-1-\lambda & \lambda+1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1-\lambda & 0 & \cdots & \lambda+1 & 0 & 0 & 0 & \cdots & 0 \\
-1 & -1 & \cdots & -1 & \lambda & -1 & -1 & \cdots & -1 \\
-2 & -2 & \cdots & -2 & -1 & \lambda & -2 & \cdots & -2 \\
0 & 0 & \cdots & 0 & 0 & -2-\lambda & \lambda+2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & -2-\lambda & 0 & \cdots & \lambda+2
\end{vmatrix}
$$

$$
= \begin{vmatrix}
\lambda-(h-2) & -1 & \cdots & -1 & -1 & -2-2(n-h-1) & -2 & \cdots & -2 \\
0 & \lambda+1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda+1 & 0 & 0 & 0 & \cdots & 0 \\
-1-(h-2) & -1 & \cdots & -1 & \lambda & -1-(n-h-1) & -1 & \cdots & -1 \\
-2-2(h-2) & -2 & \cdots & -2 & -1 & \lambda-2(n-h-1) & -2 & \cdots & -2 \\
0 & 0 & \cdots & 0 & 0 & 0 & \lambda+2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda+2
\end{vmatrix}
$$

$$= (\lambda+1)^{h-2}(\lambda+2)^{n-h-1} \begin{vmatrix} \lambda - (h-2) & -1 & -2 - 2(n-h-1) \\ -1 - (h-2) & \lambda & -1 - (n-h-1) \\ -2 - 2(h-2) & -1 & \lambda - 2(n-h-1) \end{vmatrix}$$

$$= (\lambda+1)^{h-2}(\lambda+2)^{n-h-1}[\lambda^3 + (h+4-2n)\lambda^2 + (5 - 2h - 2nh + 2h^2 - n)\lambda - nh + h^2 - 2h + 2].$$

In the following, we will prove the remaining part of Theorem 2.5. Consider the cubic function on x

$$f(x) = x^3 + (h+4-2n)x^2 + (5 - 2h - 2nh + 2h^2 - n)x - nh + h^2 - 2h + 2.$$

From a simple calculation, we have

$$\begin{cases} f(0) = -nh + h^2 - 2h + 2 = -h(n-h) - (2h-2) < 0, \\ f(-\frac{1}{2}) = \frac{3}{8} - \frac{3}{4}h < 0, \\ f(-1) = h - n + nh - h^2 = (n-h)(h-1) > 0, \\ f(-2) = 6h - 6n + 3nh - 3h^2 = (n-h)(3h-6) > 0. \end{cases}$$

Note that $f(x) \to +\infty$ $(x \to +\infty)$ and $f(0) < 0$, so there is at least one root in $(0, +\infty)$. Since $f(-\frac{1}{2}) < 0$ and $f(-1) > 0$, then there is at least one root in $(-1, -\frac{1}{2})$. By $f(x) \to -\infty$ $(x \to -\infty)$ and $f(-2) > 0$, so there is at least one root in $(-\infty, -2)$. Thus there is exactly one root in each interval. This completes the proof. $\square$

Using the similar method to compute the distance characteristic polynomials of K_n^{s+t} and $K_n^{s,t}$, we have the following two results.

Theorem 2.6 *Let $s \geq 2, t \geq 2$ and $n = s+t$. Then the distance characteristic polynomial of K_n^{s+t} is*

$$P_D(\lambda) = (\lambda+1)^{n-4}[\lambda^4 + (-s-t+4)\lambda^3 + (2t+2s-8st+4)\lambda^2 + (6s+6t-14st)\lambda - 5st + 2s + 2t].$$

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the distance spectrum of K_n^{s+t}. Then
* $\lambda_1 > 0$, $-1 < \lambda_2 < -\frac{1}{2}$ and $\lambda_3 = -1$.
* $-2 < \lambda_{n-1} < -1$ and $\lambda_n < -2$.

Proof. The distance matrix of K_n^{s+t} is

$$D = \begin{pmatrix} 0 & \cdots & 1 & 1 & 2 & 3 & \cdots & 3 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 1 & 2 & 3 & \cdots & 3 \\ 1 & \cdots & 1 & 0 & 1 & 2 & \cdots & 2 \\ 2 & \cdots & 2 & 1 & 0 & 1 & \cdots & 1 \\ 3 & \cdots & 3 & 2 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & \cdots & 3 & 2 & 1 & 1 & \cdots & 0 \end{pmatrix}.$$

Similar to the proof of Theorem 2.5, by a simple calculation, we have

$$\det(\lambda I - D) = (\lambda + 1)^{n-4} \begin{vmatrix} \lambda - (s-2) & -1 & -2 & -3 - 3(t-2) \\ -1 - (s-2) & \lambda & -1 & -2 - 2(t-2) \\ -2 - 2(s-2) & -1 & \lambda & -1 - (t-2) \\ -3 - 3(s-2) & -2 & -1 & \lambda - (t-2) \end{vmatrix}$$

$$= (\lambda + 1)^{n-4}[\lambda^4 + (-s - t + 4)\lambda^3 + (2t + 2s - 8st + 4)\lambda^2 + (6s + 6t - 14st)\lambda - 5st + 2s + 2t].$$

Consider the quartic function on x

$$f(x) = x^4 + (-s - t + 4)x^3 + (2t + 2s - 8st + 4)x^2 + (6s + 6t - 14st)x - 5st + 2s + 2t.$$

Note that $(s-1)(t-1) = st - s - t + 1 > 0$, hence $st + 1 > s + t$. Then we obtain that

$$\begin{cases} f(0) = -5st + 2s + 2t < 2(st+1) - 5st = 2 - 3st < 0, \\ f(-\frac{1}{2}) = \frac{9}{16} - \frac{3}{8}s - \frac{3}{8}t < 0, \\ f(-1) = 1 - s - t + st > 0, \\ f(-2) = 6s + 6t - 9st < 6(st+1) - 9st = 6 - 3st < 0. \end{cases}$$

Note that $f(x) \to +\infty$ $(x \to +\infty)$ and $f(0) < 0$, so there is at least one root in $(0, +\infty)$. Since $f(-\frac{1}{2}) < 0$ and $f(-1) > 0$, then there is at least one root in $(-1, -\frac{1}{2})$. Since $f(-1) > 0$ and $f(-2) < 0$, then there is at least one root in $(-2, -1)$. By $f(x) \to +\infty$ $(x \to -\infty)$ and $f(-2) < 0$, so there is at least one root in $(-\infty, -2)$. Thus there is exactly one root in each interval. The result is completed. $\square$

Theorem 2.7 *Let $s \geq 2, t \geq 2$ and $n = s + t - 1$. Then the distance characteristic polynomial of $K_n^{s,t}$ is*

$$P_D(\lambda) = (\lambda + 1)^{n-3}[\lambda^3 + (-s - t + 4)\lambda^2 + (2 + s + t - 3st)\lambda + s + t - 2st].$$

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the distance spectrum of $K_n^{s,t}$. Then
• $\lambda_1 > 0$, $-1 < \lambda_2 < -\frac{2}{3}$ and $\lambda_3 = -1$.
• $\lambda_{n-1} = -1$ and $\lambda_n < -2$.

Proof. The distance matrix of $K_n^{s,t}$ is

$$D = \begin{pmatrix} 0 & \cdots & 1 & 1 & 2 & \cdots & 2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 1 & 2 & \cdots & 2 \\ 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ 2 & \cdots & 2 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & \cdots & 2 & 1 & 1 & \cdots & 0 \end{pmatrix}.$$

Similar to the proof of Theorem 2.5, we have

$$\det(\lambda I - D) = (\lambda + 1)^{n-3} \begin{vmatrix} \lambda - (s-2) & -1 & -2 - 2(t-2) \\ -1 - (s-2) & \lambda & -1 - (t-2) \\ -2 - 2(s-2) & -1 & \lambda - (t-2) \end{vmatrix}$$

$$= (\lambda + 1)^{n-3}[\lambda^3 + (-s - t + 4)\lambda^2 + (2 + s + t - 3st)\lambda + s + t - 2st].$$

Consider the cubic function on x

$$f(x) = x^3 + (-s - t + 4)x^2 + (2 + s + t - 3st)x + s + t - 2st.$$

Note that $(s-1)(t-1) = st - s - t + 1 > 0$, then $st + 1 > s + t$. By a simple calculation, we have

$$\begin{cases} f(0) = s + t - 2st < 1 - st < 0, \\ f(-\frac{2}{3}) = \frac{4}{27} - \frac{1}{9}s - \frac{1}{9}t < 0, \\ f(-1) = 1 - s - t + st > 0. \end{cases}$$

Note that $f(x) \to +\infty$ $(x \to +\infty)$ and $f(0) < 0$, so there is at least one root in $(0, +\infty)$. Since $f(-\frac{2}{3}) < 0$ and $f(-1) > 0$, then there is at least one root in $(-1, -\frac{2}{3})$. Since $f(-1) > 0$ and $f(x) \to -\infty$ $(x \to -\infty)$, then there is at least one root in $(-\infty, -1)$. Thus there is exactly one root in each interval. This means that $\lambda_1 > 0$, $-1 < \lambda_2 < -\frac{2}{3}$, $\lambda_3 = \lambda_{n-1} = -1$ and $\lambda_n < -1$.

Obviously, the diameter of $K_n^{s,t}$ is 2, and P_3 is an induced subgraph of $K_n^{s,t}$. Moreover, $D(P_3)$ is a principal submatrix of $D(K_n^{s,t})$. It is easy to calculate that $\lambda_3(P_3) = -2$, then by Lemma 2.2, $\lambda_n(K_n^{s,t}) \leq \lambda_3(P_3) = -2$. Furthermore, $K_n^{s,t}$ is not a complete k-partite graph, then by Lemma 2.3, we have $\lambda_n < -2$. This completes the proof. $\square$

By Theorems 2.5, 2.6 and 2.7, we obtain the following corollary.

Corollary 2.8 *No two non-isomorphic graphs of K_n^h, K_n^{s+t} and $K_n^{s,t}$ are D-cospectral.*

Proof. From the distance characteristic polynomials of K_n^h, K_n^{s+t} and $K_n^{s,t}$, for any two non-isomorphic graphs belong to the same type, the result is obvious.

It is clear that K_n^{s+t} and $K_n^{s,t}$ have distinct distance spectrum, since -1 is the distance eigenvalue of K_n^{s+t} with multiplicity $n - 4$, and is the distance eigenvalue of $K_n^{s,t}$ with multiplicity $n - 3$, respectively.

Now we only need to prove that K_n^h has distinct distance spectrum with K_n^{s+t} and $K_n^{s,t}$.

Suppose that K_n^h and K_n^{s+t} are D-cospectral. Note that -1 is the distance eigenvalue of K_n^{s+t} with multiplicity $n-4$, then -1 is also the distance eigenvalue of K_n^h with multiplicity $n - 4$. On the other hand, notice that -2 is not the distance eigenvalue of K_n^{s+t}, then it follows that -2 is also not the distance eigenvalues of K_n^h, thus $n = h + 1$. Then -1 is the distance eigenvalue of K_n^h with multiplicity $n - 3$, a contradiction.

Assume that K_n^h and $K_n^{s,t}$ are D-cospectral. Note that -2 is not the distance eigenvalue of $K_n^{s,t}$, then it follows that -2 is also not the distance eigenvalue of K_n^h, so $n = h + 1$. Then we have

$$
\begin{cases}
P_{D(K_n^h)}(\lambda) = (\lambda + 1)^{n-3}[\lambda^3 + (-n + 3)\lambda^2 + (-5n + 9)\lambda - 3n + 5], \\
P_{D(K_n^{s,t})}(\lambda) = (\lambda + 1)^{n-3}[\lambda^3 + (-s - t + 4)\lambda^2 + (2 + s + t - 3st)\lambda + s + t - 2st].
\end{cases}
$$

Note that they have the same distance characteristic polynomial, hence

$$
\begin{cases}
-3n + 5 = s + t - 2st, \\
n = s + t - 1.
\end{cases}
$$

Solving the two equations we get $t = 2$ or $t = n - 1$. Hence K_n^h and $K_n^{s,t}$ are isomorphic, a contradiction. $\square$

Next, we give the distance characteristic polynomial of the friendship graph. In fact F_n^k is a special case of the graph $K_n^{n_1, n_2, \ldots, n_k}$ in [8]. If $n_1 = n_2 = \cdots = n_k = 2$, then $K_n^{n_1, n_2, \ldots, n_k} \cong F_n^k$. In [8], Liu et al. give the distance characteristic polynomial of $K_n^{n_1, n_2, \ldots, n_k}$:

$$
P_D(\lambda) = (\lambda + 1)^{n-k-1}(\lambda - \sum_{i=1}^{k} \frac{n_i(2\lambda + 1)}{\lambda + n_i + 1}) \prod_{i=1}^{k}(\lambda + n_i + 1).
$$

Hence we have the following corollary directly.

Corollary 2.9 *The distance characteristic polynomial of F_n^k is*

$$
P_D(\lambda) = (\lambda + 1)^{n-k-1}(\lambda + 3)^{k-1}[\lambda^2 - (4k - 3)\lambda - 2k].
$$

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the distance spectrum of F_n^k. Then $-1 < \lambda_2 < -\frac{1}{2}$, $\lambda_3 = -1$ and $\lambda_n = -3$.

Lemma 2.10 *If G and F_n^k have the same distance spectra, then $diam(G) = 2$ and $|E(G)| = |E(F_n^k)| = 3k$.*

Proof. Assume that $diam(G) \geq 3$, then $D(P_4)$ is a principal submatrix of $D(G)$. By Lemma 2.2, we have $\lambda_n(D(G)) \leq \lambda_4(D(P_4)) = -3.4142$, which contradicts $\lambda_n(D(G)) = -3$. Thus $diam(G) = 2$. By Lemma 2.4, $|E(G)| = |E(F_n^k)| = 3k$. $\square$

3 Main results

In this section, our first task is to show that K_n^h, K_n^{s+t} and $K_n^{s,t}$ are determined by their D-spectra. First, we give some useful graphs and their distance spectra.
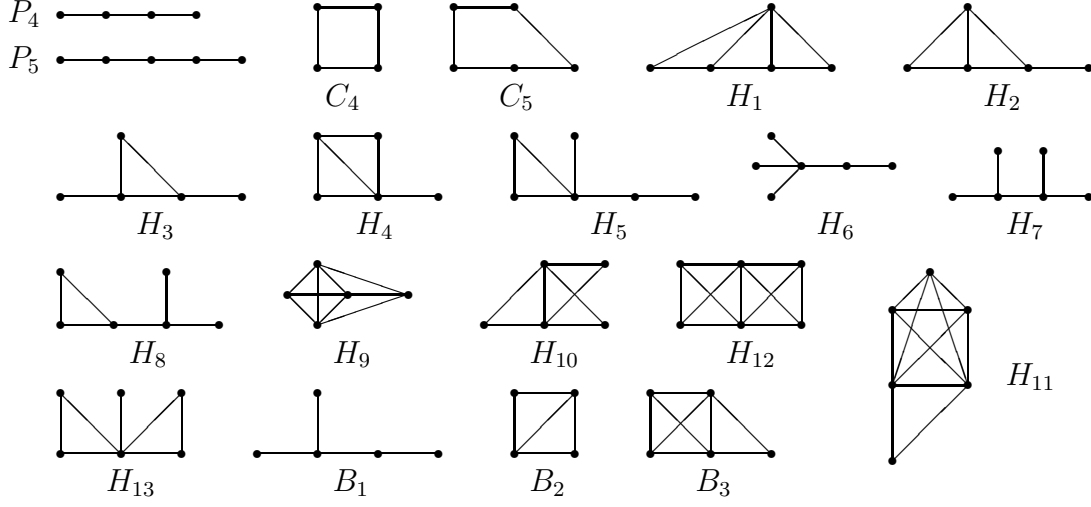
Fig. 2. Graphs P_4, P_5, C_4, C_5, $H_1 - H_{13}$ and $B_1 - B_3$.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
P_4	5.1623	-0.5858	-1.1623	-3.4142		
P_5	8.2882	-0.5578	-0.7639	-1.7304	-5.2361	
C_4	4.0000	0.0000	-2.0000	-2.0000		
C_5	6.0000	-0.3820	-0.3820	-2.6180	-2.6180	
H_1	5.2926	-0.3820	-0.7217	-1.5709	-2.6180	
H_2	6.2162	-0.4521	-1.0000	-1.1971	-3.5669	
H_3	6.6375	-0.5858	-0.8365	-1.8010	-3.4142	
H_4	5.7596	-0.5580	-0.7667	-2.0000	-2.4348	
H_5	9.3154	-0.5023	-1.0000	-1.0865	-2.3224	-4.4042
H_6	9.6702	-0.4727	-1.0566	-2.0000	-2.0000	-4.1409
H_7	10.0000	-0.4348	-1.0000	-2.0000	-2.0000	-4.5616
H_8	9.6088	-0.4931	-1.0000	-1.0924	-2.0000	-5.0233
H_9	4.4495	-0.4495	-1.0000	-1.0000	-2.0000	
H_{10}	5.3723	-0.3723	-1.0000	-2.0000	-2.0000	
H_{11}	6.1425	-0.4913	-1.0000	-1.0000	-1.0000	-2.6512
H_{12}	6.4641	-0.4641	-1.0000	-1.0000	-1.0000	-3.0000
H_{13}	7.8526	-0.6303	-1.0000	-1.0000	-2.2223	-3.0000
B_1	7.4593	-0.5120	-1.0846	-2.0000	-3.8627	
B_2	3.5616	-0.5616	-1.0000	-2.0000		
B_3	4.9018	-0.5122	-1.0000	-1.0000	-2.3896	

Next, we first show that K_n^h is determined by its D-spectrum. Let G be a graph D-cospectral to K_n^h. We call H a forbidden subgraph of G if G contains no H as an induced subgraph.

Lemma 3.1 *If G and K_n^h are D-cospectral, then C_4, C_5 and H_i ($i \in \{1, 4, 9, 10, 11, 12, 13\}$) are forbidden subgraphs of G.*

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Proof. Let G and K_n^h have the same distance spectrum. Suppose that H is an induced subgraph of G and $H \in \{C_4, C_5, H_i \ (i \in \{1, 4, 9, 10, 11, 12, 13\})\}$. Note that $\operatorname{diam}(H) = 2$, obviously $D(H)$ is a principal submatrix of $D(G)$. Let $|V(H)| = m$, by Lemma 2.2, then $\lambda_2(G) \geq \lambda_2(H)$, $\lambda_3(G) \geq \lambda_3(H)$ and $\lambda_{m-1}(H) \geq \lambda_{n-1}(G)$. By Theorem 2.5, we know that $-1 < \lambda_2(G) < -\frac{1}{2}$, $\lambda_3(G) = -1$ and $\lambda_{n-1}(G) \in \{-1, -2\}$. Hence we have $\lambda_2(H) < -\frac{1}{2}$, $\lambda_3(H) \leq -1$ and $\lambda_{m-1}(H) \geq -2$. However $\lambda_2 \geq -\frac{1}{2}$ for C_4, C_5 and $H_i \ (i \in \{1, 9, 10, 11, 12\})$; $\lambda_3 > -1$ for H_4 and $\lambda_{m-1} < -2$ for H_{13}, a contradiction. $\square$
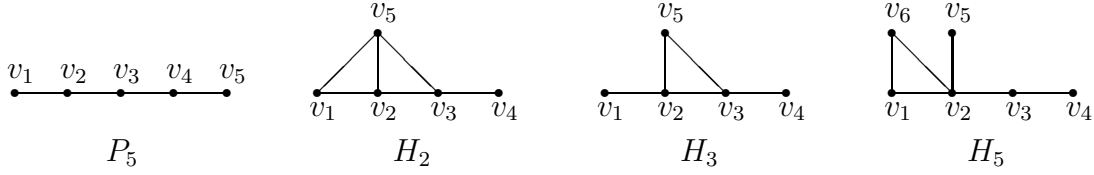

Fig. 3. The labeled graphs of P_5, H_2, H_3 and H_5.

For any $S \subseteq V(G)$, let $D_G(S)$ denote the principal submatrix of $D(G)$ obtained by S.

Lemma 3.2 *If G and K_n^h are D-cospectral, then P_5 and $H_i \ (i \in \{2, 3, 5, 6, 7, 8\})$ are forbidden subgraphs of G.*

Proof. For P_5. Suppose that P_5 is an induced subgraph of G, then $d_G(v_1, v_5) \in \{2, 3, 4\}$. If $d_G(v_1, v_5) = 4$, then $D_G(\{v_1, v_2, v_3, v_4, v_5\}) = D(P_5)$ is a principal submatrix of $D(G)$. By Lemma 2.2, we have $\lambda_3(G) \geq \lambda_3(P_5) = -0.7639 > -1$, a contradiction. If $d_G(v_1, v_5) \in \{2, 3\}$, let $d_G(v_1, v_4) = a$, $d_G(v_1, v_5) = b$ and $d_G(v_2, v_5) = c$, then $a, b, c \in \{2, 3\}$. We get the principal submatrix of $D(G)$

$$D_G(\{v_1, v_2, v_3, v_4, v_5\}) = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 1 & 0 & 1 & 2 & c \\ 2 & 1 & 0 & 1 & 2 \\ a & 2 & 1 & 0 & 1 \\ b & c & 2 & 1 & 0 \end{pmatrix}.$$

By a simple calculation, we have

(a,b,c)	$(3,3,3)$	$(3,2,2)$	$(3,2,3)$	$(3,3,2)$	$(2,3,3)$	$(2,3,2)$	$(2,2,2)$	$(2,2,3)$
λ_2	-0.4348	-0.3260	0	-0.3713	-0.3713	-0.1646	-0.2909	-0.3260

By Lemma 2.2, $\lambda_2(G) \geq \lambda_2(D_G(\{v_1, v_2, v_3, v_4, v_5\})) > -\frac{1}{2}$. Note that $\lambda_2(G) < -\frac{1}{2}$, a contradiction. Hence P_5 is a forbidden subgraph of G.

For H_2. Assume that H_2 is an induced subgraph of G, then $d_G(v_1, v_4) \in \{2, 3\}$. If $d_G(v_1, v_4) = 3$, then $D(H_2)$ is a principal submatrix of $D(G)$. By Lemma 2.2, we have $\lambda_2(G) \geq \lambda_2(H_2) = -0.4521 > -1/2$, a contradiction. If $d_G(v_1, v_4) = 2$, it is easy to calculate that $\lambda_2(D_G(\{v_1, v_2, v_3, v_4, v_5\})) = -0.2284 > -1/2$. By Lemma 2.2 and Theorem 2.5, we also get a contradiction. Therefore H_2 is a forbidden subgraph of G.

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For H_3. Suppose that H_3 is an induced subgraph of G, then $d_G(v_1, v_4) \in \{2, 3\}$. If $d_G(v_1, v_4) = 3$, then $D(H_3)$ is a principal submatrix of $D(G)$. By Lemma 2.2, we have $\lambda_3(G) \geq \lambda_3(H_3) = -0.8365 > -1$, a contradiction. If $d_G(v_1, v_4) = 2$, it is easy to check that $\lambda_2(D_G(\{v_1, v_2, v_3, v_4, v_5\})) = -0.3820 > -1/2$. By Lemma 2.2 and Theorem 2.5, we also obtain a contradiction. Hence H_3 is a forbidden subgraph of G.

For H_5. Assume that H_5 is an induced subgraph of G. If $d_G(v_1, v_4) = d_G(v_4, v_5) = d_G(v_4, v_6) = 3$, then $D(H_5)$ is a principal submatrix of $D(G)$. By Lemma 2.2, we have $\lambda_{n-1}(G) \leq \lambda_5(H_5) = -2.3224 < -2$, a contradiction. Otherwise, there exists at least one equal to 2 among $d_G(v_1, v_4), d_G(v_4, v_5)$ and $d_G(v_4, v_6)$. Without loss of generality, we may assume that $d_G(v_1, v_4) = 2$. Note that H_5 is an induced subgraph of G, then there exists a vertex $v \in V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$ such that $vv_1, vv_4 \in E(G)$. Then $G[vv_1v_2v_3v_4] = C_5$, $G[vv_1v_2v_3v_4] = H_1$, $G[vv_2v_3v_4] = C_4$ or $G[vv_1v_2v_3] = C_4$. By Lemma 3.1, C_4, C_5 and H_1 are forbidden subgraphs of G, a contradiction. Hence H_5 is a forbidden subgraph of G.

For H_6, H_7 and H_8. Suppose that they are induced subgraphs of G respectively. If $D(H_6)$, $D(H_7)$ and $D(H_8)$ are principal submatrices of $D(G)$ respectively. By Lemma 2.2, $\lambda_2(G) \geq \lambda_2(H_i) > -1/2$ where $i \in \{6, 7, 8\}$, a contradiction. Otherwise, similar to the discussion for H_5, we may also obtain the same contradictions. Thus H_6, H_7 and H_8 are forbidden subgraphs of G. $\quad\square$

Theorem 3.3 *The graph K_n^h is determined by its D-spectrum.*

Proof. When $n = h$, $h = 1$ or $h = 2$, the result is obvious. Next consider $3 \leq h \leq n-1$. Let G be a graph D-cospectral to K_n^h. By Lemma 3.2, P_5 is a forbidden graph of G, thus $\mathrm{diam}(G) \leq 3$. By $\lambda_n(G) < -2$, then $\mathrm{diam}(G) \geq 2$.

Case 1. $\mathrm{diam}(G) = 3$.

If $|V(G)| = 4$, then $G = P_4$, it is easy to check that G is not D-cospectral to K_4^3, a contradiction. Next we assume that $|V(G)| \geq 5$. Note that $\mathrm{diam}(G) = 3$, then there exists a diameter-path $P = u\tilde{u}\tilde{v}v$ with length 3 in G. Let $X = \{u, \tilde{u}, \tilde{v}, v\}$, then $G[X] = P_4$. Denote by V_i $(i = 0, 1, 2, 3, 4)$ the vertex subset of $V \setminus X$, whose each vertex is adjacent to i vertices of X. Clearly $V \setminus X = \cup_{i=0}^4 V_i$.

Claim 1. $V_4 = \emptyset$.

Suppose not, then there exists a vertex $v_4 \in V_4$ such that $G[v_4 u\tilde{u}\tilde{v}v] = H_1$, a contradiction. Hence Claim 1 holds.

Claim 2. $V_3 = \emptyset$.

Suppose not, then there exists a vertex $v_3 \in V_3$ such that v_3 is adjacent to $\{u, \tilde{u}, \tilde{v}\}$, $\{\tilde{u}, \tilde{v}, v\}$, $\{u, \tilde{u}, v\}$ or $\{u, \tilde{v}, v\}$. Then G contains an induced subgraph H_2 or C_4, a contradiction.

Let $V_2^u = \{v_2 \in V_2 | v_2 u, v_2 \tilde{u} \in E(G)\}$ and $V_2^v = \{v_2 \in V_2 | v_2 v, v_2 \tilde{v} \in E(G)\}$.

Claim 3. $V_2 = V_2^u \cup V_2^v$, $G[V_2^u]$ $(G[V_2^v]) = K_{|V_2^u|}$ $(K_{|V_2^v|})$ and $E[V_2^u, V_2^v] = \emptyset$.

For any $v_2 \in V_2$, it is impossible that v_2 is adjacent to u and v since $d_G(u,v) = 3$. If v_2 is adjacent to u and $\tilde{v}$ (or $\tilde{u}$ and v), then $G[v_2 u \tilde{u} \tilde{v}] = C_4$ (or $G[v_2 \tilde{u} \tilde{v} v] = C_4$), by Lemma 3.1, a contradiction. If v_2 is adjacent to $\tilde{u}$ and $\tilde{v}$, then $G[v_2 u \tilde{u} \tilde{v} v] = H_3$, a contradiction. Thus $V_2 = V_2^u \cup V_2^v$. For any $v_2, v_2^\star \in V_2^u$, then $v_2 v_2^\star \in E(G)$. Otherwise $G[v_2 v_2^\star u \tilde{u} \tilde{v}] = H_4$, a contradiction. This means that $G[V_2^u] = K_{|V_2^u|}$. Similarly, $G[V_2^v] = K_{|V_2^v|}$. If $v_2 v_2^\star \in E(G)$ for any $v_2 \in V_2^u$ and $v_2^\star \in V_2^v$, then $G[v_2 v_2^\star \tilde{u} \tilde{v}] = C_4$, a contradiction. Hence $E[V_2^u, V_2^v] = \emptyset$.

Claim 4. $|V_1| \leq 1$.

Let $v_1 \in V_1$. Obviously, v_1 can only be adjacent to $\tilde{u}$ or $\tilde{v}$, otherwise $G[v_1 u \tilde{u} \tilde{v} v] = P_5$, a contradiction. Now we assume that $|V_1| \geq 2$. Let $v_1, v_1^\star \in V_1$. If they are adjacent to the same vertex of X, then $G[v_1 v_1^\star u \tilde{u} \tilde{v} v] = H_5$ or H_6, a contradiction. Otherwise, $G[v_1 v_1^\star u \tilde{u} \tilde{v} v] = H_7$ or $G[v_1 v_1^\star \tilde{u} \tilde{v}] = C_4$, a contradiction. Hence Claim 4 is completed.

Claim 5. Only one set is nonempty between V_1 and V_2.

Suppose not, then there exist two vertices $v_1 \in V_1$ and $v_2 \in V_2$. Without loss of generality, we may assume that v_2 is adjacent to u and $\tilde{u}$. If v_1 is adjacent to $\tilde{u}$, then $G[v_1 v_2 u \tilde{u} \tilde{v} v] = H_5$ or $G[v_1 v_2 u \tilde{u} \tilde{v}] = H_4$, a contradiction. If v_1 is adjacent to $\tilde{v}$, then $G[v_1 v_2 u \tilde{u} \tilde{v} v] = H_8$ or $G[v_1 v_2 \tilde{u} \tilde{v}] = C_4$, a contradiction. Thus Claim 5 holds.

Claim 6. $V_0 = \emptyset$.

Suppose not, then there exist a vertex $v_0 \in V_0$ such that $v_0 v^\star \in E(G)$, where $v^\star \in V_1 \cup V_2$. Then $G[v_0 v^\star \tilde{u} \tilde{v} v] = P_5$ or $G[v_0 v^\star u \tilde{u} \tilde{v}] = P_5$, a contradiction.

By Claims 1-6, we have $V = V_1 \cup V_2 \cup X$. If $|V_1| = 1$, then by Claim 5, $V_2 = \emptyset$. This means that $G \cong B_1$. It is easy to check that B_1 has distinct D-spectrum with K_5^h, a contradiction. So we have $V_1 = \emptyset$, then $V_2 \neq \emptyset$, and thus $G \cong K_n^{s+t}$. By Corollary 2.8, K_n^{s+t} has distinct D-spectrum with K_n^h, a contradiction. It follows that there is no graph G with diameter 3 D-cospectral to K_n^h.

Case 2. $\operatorname{diam}(G) = 2$.

There exists a diameter-path $P = xyz$ with length 2 in G. Let $X = \{x, y, z\}$, then $G[X] = P_3$. Obviously, $V \backslash X \neq \emptyset$ since $n \geq 4$. Denote by V_i ($i = 0, 1, 2, 3$) the vertex subset of $V \backslash X$, whose each vertex is adjacent to i vertices of X. Clearly $V \backslash X = \cup_{i=0}^3 V_i$.

Claim 7. $|V_3| \leq 1$.

Suppose not, there exist two vertices $v_3, v_3^\star \in V_3$. If $v_3 v_3^\star \in E(G)$, $G[v_3 v_3^\star xyz] = H_9$, a contradiction. Otherwise $v_3 v_3^\star \notin E(G)$, then $G[v_3 v_3^\star xz] = C_4$, a contradiction. Therefore Claim 7 holds.

Let $V_{xy} = \{v_2 \in V_2 | v_2 x, v_2 y \in E(G)\}$, $V_{yz} = \{v_2 \in V_2 | v_2 y, v_2 z \in E(G)\}$.

Claim 8. $V_2 = V_{xy} \cup V_{yz}$, $G[V_{xy}]$ $(G[V_{yz}]) = K_{|V_{xy}|}$ $(K_{|V_{yz}|})$, and $E[V_{xy}, V_{yz}] = \emptyset$.

For any $v_2 \in V_2$, it is impossible that v_2 is adjacent to x and z since $G[v_2 xyz] = C_4$. Hence $V_2 = V_{xy} \cup V_{yz}$. For any $v_2, v_2^\star \in V_{xy}$, then $v_2 v_2^\star \in E(G)$. Otherwise $G[v_2 v_2^\star xyz] = H_4$, a contradiction. This means that $G[V_{xy}] = K_{|V_{xy}|}$. Similarly, $G[V_{yz}] = K_{|V_{yz}|}$. If

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$E[V_{xy}, V_{yz}] \neq \emptyset$, then there exist two vertices $v_2 \in V_{xy}$ and $v_2^\star \in V_{yz}$ such that $v_2 v_2^\star \in E(G)$, and thus $G[v_2 v_2^\star xyz] = H_1$, a contradiction. Hence $E[V_{xy}, V_{yz}] = \emptyset$.

Claim 9. If $v_1 \in V_1$, then v_1 must be adjacent to y.

Suppose not, then v_1 is adjacent to x or z. Without loss of generality, we may assume that $v_1 x \in E(G)$. Note that $\mathrm{diam}(G) = 2$, then there exists a vertex $u \in V \backslash X$ such that $uv_1, uz \in E(G)$, and thus $u \in \cup_{i=1}^3 V_i$. If $u \in V_1$, then $G[uv_1 xyz] = C_5$, a contradiction. If $u \in V_2$, by Claim 8, u is adjacent to y and z, and then $G[uv_1 xy] = C_4$, a contradiction. If $u \in V_3$, then $G[uv_1 xyz] = H_1$, a contradiction. Thus Claim 9 holds.

Claim 10. $V_0 = \emptyset$.

Suppose not, then there exists a vertex $v_0 \in V_0$ such that v_0 is adjacent to some vertices of $V_1 \cup V_2 \cup V_3$. If v_0 is adjacent to only one vertex u of $V_1 \cup V_2 \cup V_3$, then $u \in V_3$ since $\mathrm{diam}(G) = 2$, and thus $G[v_0 uxyz] = H_4$, a contradiction. So v_0 must be adjacent to at least two vertices of $V_1 \cup V_2 \cup V_3$, we always find an induced subgraph C_4 of G at each case, a a contradiction. Therefore Claim 10 is obtained.

By Claim 10, $\emptyset \neq V \backslash X = \cup_{i=1}^3 V_i$. Next we distinguish the following four cases.

Subcase 2.1. $V_3 \neq \emptyset$.

By Claim 7, $|V_3| = 1$. Note that H_4 and H_{10} are forbidden subgraphs of G, then $V_1 = \emptyset$. Let $V_3 = \{v_3\}$. Obviously, $v_2 v_3 \in E(G)$ for each $v_2 \in V_2$. Otherwise $G[v_2 v_3 xyz] = H_1$, a contradiction. If $|V_2| \leq 2$, i.e., there exist two vertices $v_2, v_2^\star \in V_2$, then $G[v_2 v_2^\star v_3 xyz] = H_{11}$ or H_{12}, a contradiction. So we have $|V_2| \leq 1$. If $V_2 = \emptyset$, then $G \cong B_2$, it is easy to check that B_2 has distinct distance spectrum with K_4^3, a contradiction. If $|V_2| = 1$, then $G \cong B_3$. Clearly, B_3 is not D-cospectral to K_5^h, a contradiction.

Subcase 2.2. $V_3 = \emptyset$, $V_2 \neq \emptyset$ and $V_1 = \emptyset$.

By Claim 8, $G \cong K_n^{n-1}$ or $G \cong K_n^{s,t}$. By Corollary 2.8, $K_n^{s,t}$ and K_n^h have distinct distance spectra, a contradiction. Then $G \cong K_n^{n-1}$.

Subcase 2.3. $V_3 = \emptyset$, $V_2 \neq \emptyset$ and $V_1 \neq \emptyset$.

For any $v_1 \in V_1$, we claim that $d(v_1) = 1$. In fact, if $d(v_1) \geq 2$, then there exists a vertex $v_2 \in V_2$ such that $v_1 v_2 \in E(G)$, and then $G[v_1 v_2 xyz] = H_4$, a contradiction. Furthermore, we claim that only one set is nonempty between V_{xy} and V_{yz}. Otherwise, let $v_2 \in V_{xy}$ and $v_2^\star \in V_{yz}$, then $G[v_2 v_2^\star xyz] = H_{13}$, a contradiction. Hence $G \cong K_n^h$.

Subcase 2.4. $V_3 = \emptyset$, $V_2 = \emptyset$ and $V_1 \neq \emptyset$.

Let $V_1^\star = \{v \in V_1 | d(v) \geq 2\}$. If $V_1^\star = \emptyset$, then $G \cong K_{1,n-1}$. Note that $\lambda_n(K_{1,n-1}) = -2$, then $K_{1,n-1}$ is not D-cospectral to K_n^h, a contradiction. If $V_1^\star \neq \emptyset$, we claim that $G[V_1^\star] = K_{|V_1^\star|}$. If not, there exist $u, v \in V_1^\star$ such that $uv \notin E(G)$. If there exists a vertex $w \in V_1^\star$ such that $wu, wv \in E(G)$, then $G[wuvxy] = H_4$, a contradiction. Otherwise, there exist two distinct vertices $w_1 \in V_1^\star$ and $w_2 \in V_1^\star$ such that $w_1 u \in E(G)$ and $w_2 v \in E(G)$, then $w_1 w_2 \in E(G)$ since H_{13} is a forbidden subgraph of G. Thus $G[w_1 w_2 uvy] = H_1$, a contradiction. Hence $G[V_1^\star] = K_{|V_1^\star|}$, it means that $G \cong K_n^h$.

Therefore, if G is a graph D-cospectral to K_n^h, then $G \cong K_n^h$. This completes the proof of Theorem 3.3. $\square$

Theorem 3.4 *The graph K_n^{s+t} is determined by its D-spectrum.*

Proof. Let G be a graph D-cospectral to K_n^{s+t}. From Theorem 2.6, we know that $-1 < \lambda_2(G) < -\frac{1}{2}$, $\lambda_3(G) = -1$ and $-2 < \lambda_{n-1}(G) < -1$. Similar to the proof of Lemmas 3.1 and 3.2, we also get P_5, C_4, C_5 and H_i $(i = 1, 2, \ldots, 13)$ are forbidden subgraphs of G. Note that P_5 is a forbidden subgraph of G and $\lambda_n(G) < -2$, then $2 \le \mathrm{diam}(G) \le 3$. By the above forbidden subgraphs, similar to the proof of Theorem 3.3, we have:

- If $\mathrm{diam}(G) = 3$, then $G \cong B_1$ or $G \cong K_n^{s+t}$.

- If $\mathrm{diam}(G) = 2$, then $G \cong B_2$, $G \cong B_3$, $G \cong K_n^h$ or $G \cong K_n^{s,t}$.

From D-spectra of B_i $(i = 1, 2, 3)$ and Corollary 2.8, then we must have $G \cong K_n^{s+t}$. Thus the theorem follows. $\square$

Theorem 3.5 *The graph $K_n^{s,t}$ is determined by its D-spectrum.*

Proof. Let G be a graph D-cospectral to $K_n^{s,t}$. By Theorem 2.7, then $-1 < \lambda_2(G) < -\frac{2}{3} < -\frac{1}{2}$, $\lambda_3(G) = \lambda_{n-1}(G) = -1$. Hence we can still use P_5, C_4, C_5 and H_i $(i = 1, 2, \ldots, 13)$ as the forbidden subgraph of G. Note that P_5 is a forbidden subgraph of G and $\lambda_n(G) < -2$, then $2 \le \mathrm{diam}(G) \le 3$. Similar to the proof of Theorem 3.3, then

- If $\mathrm{diam}(G) = 3$, then $G \cong B_1$ or $G \cong K_n^{s+t}$.

- If $\mathrm{diam}(G) = 2$, then $G \cong B_2$, $G \cong B_3$, $G \cong K_n^h$ or $G \cong K_n^{s,t}$.

By D-spectra of B_i $(i = 1, 2, 3)$ and Corollary 2.8, then $G \cong K_n^{s,t}$. Thus $K_n^{s,t}$ is determined by its D-spectrum. $\square$

Next, we will show that the friendship graph F_n^k is determined by its D-spectrum. In [8], Liu et al. proved that the graphs with $\lambda_2(D(G)) \le \frac{17-\sqrt{329}}{2} \approx -0.5692$ are determined by their D-spectra. Note that $\lambda_2(D(F_n^k)) < \frac{17-\sqrt{329}}{2}$ for $k \le 4$. Thus we only need to prove that F_n^k is determined by its D-spectrum when $k \ge 5$.
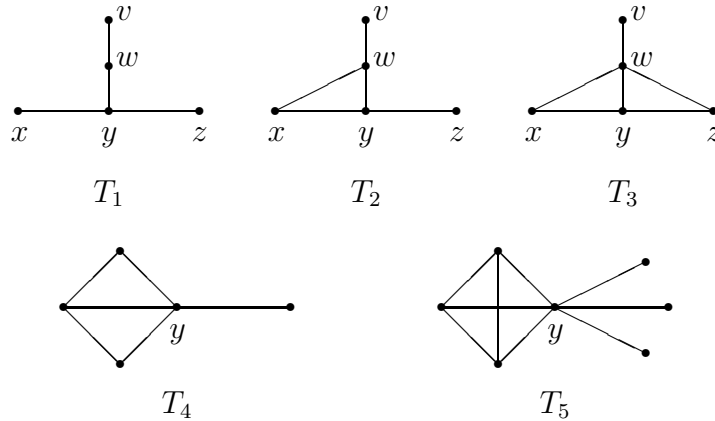

Fig. 4. Graphs $T_1 - T_5$.

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Theorem 3.6 *The friendship graph F_n^k is determined by its D-spectrum.*

Proof. Let G be a graph D-cospectral to F_n^k. By Lemma 2.10, $\operatorname{diam}(G) = 2$ and $|E(G)| = 3k$. Let $P = xyz$ be a diameter path of G.

Claim 1. $d_G(y) = n - 1 = 2k$.

If there exists a vertex $v \in V(G)$ such that $vy \notin E(G)$, then $d_G(v, y) = 2$, thus

$$
D_G(\{x, y, z, v\}) = \begin{pmatrix} 0 & 1 & 2 & a \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & b \\ a & 2 & b & 0 \end{pmatrix}.
$$

Then $a, b \in \{1, 2\}$, by a simple calculation, we have

(a, b)	$(1, 1)$	$(1, 2)$	$(2, 1)$	$(2, 2)$
λ_2	0.0000	-0.3820	-0.3820	-0.6519

By Lemma 2.2, only the case $a = 2, b = 2$ satisfies $\lambda_2(D(G)) < -\frac{1}{2}$. Thus there exists a vertex w such that the subgraph of G induced by vertices v, w, x, y, z is T_1, T_2 or T_3 (see Fig. 4). We get a principal submatrix of $D(G)$ for each case.

$$
D_1 = \begin{pmatrix} 0 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 2 \\ 2 & 1 & 2 & 0 & 1 \\ 2 & 2 & 2 & 1 & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 2 & 1 & 0 \end{pmatrix}, D_3 = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 1 & 0 \end{pmatrix}.
$$

By a simple calculation, we have $\lambda_2(D_1) = -0.2248$, $\lambda_2(D_2) = -0.3820$ and $\lambda_3(D_3) = -0.7667$. For each case, by Lemma 2.2, we have $\lambda_2(D(G)) \geq \lambda_2(D_1) = -0.2248$, $\lambda_2(D(G)) \geq \lambda_2(D_2) = -0.3820$ and $\lambda_3(D(G)) \geq \lambda_3(D_3) = -0.7667$, a contradiction. Thus Claim 1 holds.

Claim 1 implies that there are k edges in $G - y$. If $G - y$ has s connected components, then $k = |E(G - y)| \geq 2k - s$, hence $s \geq k \geq 5$.

Let $V_1, V_2, \ldots, V_s$ be the vertex sets of the s components of $G - y$.

Claim 2. $|V_i| \leq 2$ for $i = 1, 2, \ldots, s$.

Assume that there exists a vertex set V_i such that $|V_i| \geq 3$. If $G[V_i]$ is not a complete graph. Choosing three continuous vertices on the diameter path of $G[V_i]$, a vertex in another component, and vertex y. The induced subgraph of these five vertices is T_4 (see Fig. 4). Note that $\lambda_3(D(T_4)) = -0.7767$, by Lemma 2.2, $\lambda_3(D(G)) \geq \lambda_3(D(T_4)) = -0.7667$, a contradiction. If $G[V_i]$ is a complete graph. Choosing three vertices in V_i, one vertex in three other components respectively, and vertex y. The induced subgraph of these seven vertices is T_5 (see Fig. 4). Note that $\lambda_7(D(T_5)) = -3.0984$, by Lemma 2.2, $\lambda_n(D(G)) \leq \lambda_7(D(T_5)) = -3.0984$, a contradiction. Hence Claim 2 holds.

Claim 2 implies that there is at most one edge in each component. Since $|E(G-y)| = k$ and $|V(G-y)| = 2k$, it follows that $s = k$ and $G[V_i] = K_2$ for $i = 1, 2, \ldots, k$. Thus $G \cong F_n^k$. This complete the proof. $\square$

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